Asian Journal of Statistical Sciences
Vol. 5, No-1, (2025), pp. 57-69

ARF INDIA
Academic Open Access Publishing
Journal homepage link:
http://arfjournals.com/ajss

SHRINKAGE BASED ESTIMATION FOR STRESS STRENGTH RELIABILITY $P[Y < X < Z]$ FOR THE EXPONENTIAL DISTRIBUTION

Jeevanand E S[a], Anjana E. J[b], Sowbhagya S Prabhu[c] and Joseph Justin Rebello[d]

[a]Department of Statistics and Data Science, Christ University, Karnataka, India
[b]Department of Mathematics, Rheinische Friedrich Wilhelms Universität, Bonn 53115, Germany
[c]Department of Statistics, Rajagiri College of Social Sciences, Kerala, India
[d]St. Albert's College (Autonomous), Ernakulam 682018, Kerala, India

ARTICLE HISTORY
Compiled September 26, 2025

Received 27 March 2025; Accepted 17 July 2025

ABSTRACT
Estimating stress-strength reliability of the form $P(Y < X < Z)$ is a pivotal concern in reliability analysis, particularly when systems are subjected to both lower and upper stress limits. This paper investigates the reliability measure under the assumption that the stress variables Y and Z, as well as the strength variable X, follow exponential distributions. The analysis is conducted using both complete samples and right-censored samples to reflect realistic data collection scenarios. To improve estimation efficiency, we propose several shrinkage estimators based on distinct strategies: a constant shrinkage weight factor, a modified Thompson-type shrinkage weight, and the formulation introduced by Mehta and Srinivasan (1971). The performance of these estimators is evaluated via extensive Monte Carlo simulations and compared against the conventional maximum likelihood estimator, demonstrating the relative merits and limitations of each approach.

KEYWORDS
Exponential distribution, Shrinkage estimation, Stress Strength reliability.

1. Introduction

As our daily activities and businesses increasingly rely on information technology, system reliability has become a critical factor in today's world. Reliability originated in the latter half of the 20th century and was initially limited to the probability of non-failure of sophisticated equipment within a specific period.

One important aspect of system reliability is stress-strength reliability, which determines the probability of equipment failure when the variable's stress exceeds the strength. The stress-strength model has found numerous applications in engineering, psychology, pedagogy, pharmaceuticals, and many other fields, as highlighted in Johnson's (1988) review paper and Kotz, Lumelskii, and Pensky's (2003) book.

Exponential distribution is a reliable model for an item or product's life, regardless of its age, due to its constant hazard rate and lack of memory property. The classical stress strength reliability model $P[X < Y]$, where X represents stress and Y represents strength for an exponential

CONTACT Jeevanand E S[a]. Email: jeevanand.es@christuniversity.in. ORCID: https://orcid.org/0000-0002-7102-4179
CONTACT Anjana E. J[b]. Email: anjanajeevanand@gmail.com. ORCID: https://orcid.org/0000-0001-6072-8886
CONTACT Sowbhagya S Prabhu[c]. Email: sowbhagya@rajagiri.edu. ORCID: https://orcid.org/0000-0002-1892-141X
CONTACT Joseph Justin Rebello[d]. Email: jjrebello@alberts.edu.in. ORCID: https://orcid.org/0009-0009-7361-8147

distribution, has been extensively studied by numerous researchers such as Tong (1974, 1975), Johnson (1975), Kelly et al. (1976), Bai and Hong (1992), and Jeevanand and Nair (1994). Recent advances in the estimation of $R = P[X < Y]$ when X and Y follow the exponential distribution under various data situations have also been discussed by Krishnamoorthy et al. (2007), Baklizi (2013), Salehi Mahdi et al. (2015), and Safariyan et al. (2019).

In stress strength reliability studies, the reliability measure $R = P[Y < X < Z]$ involving two stress variables is a natural extension and has significant importance. Extremes in stress levels, whether excessively high or abnormally low, can compromise the structural integrity or functional efficacy of systems and organisms. For example, electrical appliances such as refrigerators may malfunction under conditions of voltage deviation, while human physiological systems exhibit vulnerability when blood pressure deviates beyond normative systolic and diastolic thresholds. To illustrate the practical significance of the stress-strength reliability model $P(Y < X < Z)$, consider the thermal regulation of a medical cold storage unit used for storing sensitive vaccines. The functionality of the unit depends critically on maintaining internal temperature within a prescribed range—say, between 2°C and 8°C. Here, the lower and upper bounds (2°C and 8°C) represent stress limits Y and Z, while the system's actual cooling performance X must remain strictly within these bounds to preserve the vaccines efficacy. If the temperature falls below 2°C or rises above 8°C, the system fails in its intended purpose. Thus, the model $P(Y < X < Z)$ reflects the probability that the unit's performance is both not too weak and not excessively strong—a dual threshold that is vital in applications where overperformance can be as detrimental as underperformance. Similar logic applies to blood pressure monitoring, voltage regulation in sensitive electronics, and structural load scenarios in civil engineering, where both excessive and insufficient stress can lead to failure.

Such examples reinforce the need for estimating reliability within a two-sided stress constraint framework, highlighting the real-world applicability of the proposed methodology. The quantification and regulation of such stress boundaries hold significant relevance across diverse domains, including engineering design, psychological assessment, genetic research, and biomedical applications. The estimation of $R = P[X < Y < Z]$ has been studied by several authors. Singh (1980) discussed the Maximum Likelihood and the Empirical Estimator of $R = P[X < Y < Z]$, while Dutta and Srivastav (1986) estimated R for exponential distribution variables. Ivshin (1998) studied the Uniformly Minimum Variance Unbiased Estimate of R for variables Y, Z, and X under uniform or exponential distribution. Hameed et al. (2020) discussed the estimation of $R = P[Y < X < Z]$ when the stress and strength variables follow Kumaraswamy Distribution, and Nada and Ali (2021) discussed the estimation of Stress-Strength Reliability using Dagum Distribution. Anjana (2025) and Swapna et.al. (2025) discussed the Estimation problem of $R = P[Y < X < Z]$ when the underlying distribution is exponential and Neethu and Anjana (2023a) to the Lomax distribution.

A shrinkage estimator is an estimate that is obtained by shrinking a raw estimate. The shrinkage estimate for the population mean was proposed by Thompson (1968) and Mehta and Srinivasan (1971). Abu-Salih et al. (1988) found the shrinkage estimate of the parameters of the exponential distribution. Siu-Keung Tse et al. (1996) and Hojatollah Zakerzadeh (2016) discussed the shrinkage estimation of reliability for exponentially distributed lifetimes, while Mehdi Jabbari Nooghabi (2016) obtained the shrinkage estimation of $P(Y < X)$ in the exponential distribution mixed with an exponential distribution. Glifin Francis et al. (2022) obtained the shrinkage estimator of stress strength reliability $R = P[X < Y]$ when X and Y are geometrically distributed, using record values. In recent studies, Neethu and Anjana (2023) extended the estimation of stress–strength reliability to the case of the Lomax distribution.

Although the exponential distribution is widely used in life testing and stress strength reliability, there appears to be a lack of discussion in the literature on the shrinkage estimation of $R = P[Y < X < Z]$ when this distribution is employed. This study represents a modest attempt to address this gap. The paper is structured as follows: Section 3 will discuss shrinkage estimation under a complete sample. Section 4 will cover shrinkage estimates under a type II censored sample. Lastly, section 5 presents simulation studies, and the paper concludes with our findings.

2. Preliminary

Let X be the strength of the random variable following an exponential distribution with parameters λ, and Y and Z be the sress of the random variable following an exponential distribution with parameter θ and α, the corresponding probability density functions are given below.

$$f(x,\lambda) = \lambda e^{-\lambda x} \quad x > 0, \tag{1}$$

$$f(y,\theta) = \theta e^{-\theta y} \quad y > 0, \tag{2}$$

$$f(z,\alpha) = \alpha e^{-\alpha z} \quad z > 0. \tag{3}$$

Under this situation the stress strength reliability is obtained as

$$R = P[X < Y < Z] = \int_o^\infty F_y(x)(1 - F_z(x))f(x)dx = \frac{\lambda\theta}{(\alpha+\theta+\lambda)(\alpha+\lambda)}; \; 0 < R < 1. \tag{4}$$

3. Shrinkage Estimation Under Complete Sample

Let $\underline{x} = (x_1, x_2, \ldots, x_n)$ be the random sample of n observations taken from exponential distribution (1), then its likelihood function is given by

$$L(\lambda|\underline{x}) = \lambda^n e^{-\lambda n\overline{x}}; \quad \overline{x} = \frac{1}{n}\sum x_i. \tag{5}$$

Let $\underline{y} = (y_1, y_2, \ldots, y_m)$ be the random sample of m observations taken from exponential distribution (2), then its likelihood function is given by

$$L(\theta|\underline{y}) = \theta^m e^{-\theta m\overline{y}}; \quad \overline{y} = \frac{1}{m}\sum y_i. \tag{6}$$

Let $\underline{z} = (z_1, z_2, \ldots, z_k)$ be the random sample of k observations taken from exponential distribution (3), then its likelihood function is given by

$$L(\alpha|\underline{z}) = \alpha^k e^{-\alpha k\overline{z}}; \quad \overline{z} = \frac{1}{k}\sum z_i. \tag{7}$$

The joint likelihood function is given by

$$L(\lambda,\theta,\alpha|\underline{x},\underline{y},\underline{z}) = \lambda^n\theta^m\alpha^k e^{-[\lambda n\overline{x}+\theta m\overline{y}+\alpha k\overline{z}]}. \tag{8}$$

Taking the logarithm and differentiating with respect to λ, θ,and α we get the mle of λ, θ,and α as

$$\hat{\lambda}_{mle} = \frac{1}{\overline{x}}, \tag{9}$$

$$\hat{\theta}_{mle} = \frac{1}{\overline{y}}, \tag{10}$$

$$\hat{\alpha}_{mle} = \frac{1}{\overline{z}}. \tag{11}$$

Substituting this in (4), we derive the mle of R as

$$\hat{R}_{mle} = \frac{\hat{\lambda}_{mle}\hat{\theta}_{mle}}{(\hat{\alpha}_{mle}+\hat{\theta}_{mle}+\hat{\lambda}_{mle})(\hat{\alpha}_{mle}+\hat{\lambda}_{mle})} = \frac{\overline{xz}^2}{(\overline{x}\,\overline{y}+\overline{x}\,\overline{z}+\overline{y}\,\overline{z})(\overline{x}+\overline{z})} \tag{12}$$

From the above expression, it is very difficult to find the exact variance and distribution of $\hat{R}_{mle}$. Therefore, we use the multivariate delta method (See Wasserman, (2003), Soliman et al. (2013), Dhanya, and Jeevavand, (2018), Khan, and Khatoon, (2019)) to find the approximate estimate of the asymptotic variance of $\hat{R}_{mle}$ which is given as

Let the Fisher Information matrix ϕ

$$\phi(\alpha_1,\alpha_2,\alpha_3) = \begin{bmatrix} \mathbb{E}\left(-\frac{\partial^2 \ln L}{\partial\lambda^2}\right) & \mathbb{E}\left(-\frac{\partial^2 \ln L}{\partial\lambda\,\partial\theta}\right) & \mathbb{E}\left(-\frac{\partial^2 \ln L}{\partial\lambda\,\partial\alpha}\right) \\ \mathbb{E}\left(-\frac{\partial^2 \ln L}{\partial\lambda\,\partial\theta}\right) & \mathbb{E}\left(-\frac{\partial^2 \ln L}{\partial\theta^2}\right) & \mathbb{E}\left(-\frac{\partial^2 \ln L}{\partial\theta\,\partial\alpha}\right) \\ \mathbb{E}\left(-\frac{\partial^2 \ln L}{\partial\lambda\,\partial\alpha}\right) & \mathbb{E}\left(-\frac{\partial^2 \ln L}{\partial\theta\,\partial\alpha}\right) & \mathbb{E}\left(-\frac{\partial^2 \ln L}{\partial\alpha^2}\right) \end{bmatrix} \tag{13}$$

and

$$B' = \left[\ \frac{\partial R}{\partial\lambda} \quad \frac{\partial R}{\partial\theta} \quad \frac{\partial R}{\partial\alpha}\ \right] = \left[\ b1 \quad b2 \quad b3\ \right]. \tag{14}$$

Then $\sigma^2_R = V(R) = B'\phi^{-1}B$.
In this case

$$\phi(\lambda,\theta,\alpha) = \begin{bmatrix} \frac{n}{\lambda^2} & 0 & 0 \\ 0 & \frac{m}{\theta^2} & 0 \\ 0 & 0 & \frac{k}{\alpha^2} \end{bmatrix} \tag{15}$$

and

$$\phi^{-1}(\lambda,\theta,\alpha) = \begin{bmatrix} \frac{\lambda^2}{n} & 0 & 0 \\ 0 & \frac{\theta^2}{m} & 0 \\ 0 & 0 & \frac{\alpha^2}{k} \end{bmatrix}. \tag{16}$$

Also

$$b_1 = \frac{\partial R}{\partial\lambda} = \frac{\theta(\theta\alpha+\alpha^2-\lambda^2)}{(\theta+\alpha+\lambda)^2(\alpha+\lambda)^2}, \tag{17}$$

$$b_2 = \frac{\partial R}{\partial\theta} = \frac{\lambda}{(\lambda+\alpha+\theta)^2}, \tag{18}$$

$$b_3 = \frac{\partial R}{\partial\alpha} = \frac{\lambda\theta(2\alpha+2\lambda+\theta)}{(\lambda+\alpha+\theta)^2(\alpha+\lambda)^2}. \tag{19}$$

Then

$$\sigma^2_{Rmle} = V(R) = B'\phi^{-1}B = \frac{b_1^2\lambda^2}{n} + \frac{b_2^2\theta^2}{m} + \frac{b_3^2\alpha^2}{k}. \tag{20}$$

By replacing the parameters with their maximum likelihood estimate, we get the estimate $\hat{\sigma}^2_{Rmle}$ of σ^2_{Rmle}.

3.1. Shrinkage Estimation with Constant Shrinkage Factor

In this case we obtain the shrinkage estimate,

$$\hat{\beta}_{sh} = \psi(\hat{\beta})\hat{\beta}_{ub} + (1-\psi(\hat{\beta}))\hat{\beta}_0$$

with $\psi(\hat{\beta}) = 0.01$ the constant shrinkage weight factor suggested by Hameed et.al. (2020) gives the Shrinkage estimates of λ, θ and α as

$$\hat{\lambda}_{sh} = 0.01\hat{\lambda}_{ub} + 0.99\hat{\lambda}_0, \tag{21}$$

$$\hat{\theta}_{sh} = 0.01\hat{\theta}_{ub} + 0.99\hat{\theta}_0 \tag{22}$$

and

$$\hat{\alpha}_{sh} = 0.01\hat{\alpha}_{ub} + 0.99\hat{\alpha}_0. \tag{23}$$

where $\hat{\lambda}_{ub} = \frac{n-1}{n\bar{x}}$, $\hat{\theta}_{ub} = \frac{m-1}{m\bar{y}}$ and $\hat{\alpha}_{ub} = \frac{k-1}{k\bar{z}}$. $\hat{\lambda}_0$, $\hat{\theta}_0$ and $\hat{\alpha}_0$ is taken as the boot strap estimate of λ, θ and α.
This leads to the constant shrinkage weight factor of R as

$$\hat{R}_{sh} = \frac{\hat{\lambda}_{sh}\hat{\theta}_{sh}}{(\hat{\alpha}_{sh} + \hat{\theta}_{sh} + \hat{\lambda}_{sh})(\hat{\alpha}_{sh} + \hat{\lambda}_{sh})}. \tag{24}$$

3.2. The modified Thompson-type shrinkage estimator

Here we use two types of shrinkage estimates, the first one the modified Thompson-type shrinkage weight factor, and the Shrinkage weight factor suggested by Mehta & Srinivasan (1971), to find out the shrinkage estimator.

(a) Suggested by Hameed et.al. (2020) here we take the weight factor as

$$\phi(\hat{R}) = \frac{\hat{R}_{ub} - \hat{R}_0}{(\hat{R}_{ub} - \hat{R}_0)^2 + \text{var}(\hat{R}_{ub})}(0.0001), \tag{25}$$

where $\hat{R}_{ub} = \frac{\hat{\lambda}_{ub}\hat{\theta}_{ub}}{(\hat{\alpha}_{ub}+\hat{\theta}_{ub}+\hat{\lambda}_{ub})(\hat{\alpha}_{ub}+\hat{\lambda}_{ub})}$ and $\text{var}(\hat{R}_{ub})$ is defined in (20). So the modified Thomason type shrinkage estimator will be

$$\hat{R}_{Th} = \phi(\hat{R})\hat{R}_{ub} + (1 - \phi(\hat{R}))\hat{R}_0. \tag{26}$$

(b) Shrinkage weight factor suggested by Mehta & Srinivasan (1971) here we take the weight factor as

$$\phi(\hat{R}) = a \cdot \exp\left\{-\frac{b(\hat{R}_{ub} - \hat{R}_0)^2}{\text{var}(\hat{R}_{ub})}\right\}, \tag{27}$$

where $0 < a < 1$ and $b > 0$. So the Mehta & Srinivasan type shrinkage estimator will be

$$\hat{R}_{ms} = \phi(\hat{R})\hat{R}_{ub} + (1 - \phi(\hat{R}))\hat{R}_0. \tag{28}$$

4. Shrinkage Estimation of R based on right-censored sample

Here we obtain the Shrinkage Estimation of R when the Strength X is under a right censored sample.

Let $\underline{x} = (x_1, x_2, \ldots, x_{(n-p)})$ be the right censored sample with p observations censored from right from the exponential distribution (1).

The likelihood function with respect to right censored sample is given by

$$L(\underline{x}|\lambda) = [1 - F(x_{(n-p)})]^p \prod_{i=1}^{n-p} f(x_i). \tag{29}$$

Substituting (1) in (29) we get the likelihood function of right censored sample as

$$L(\underline{x}|\lambda) = \lambda^{(n-p)} e^{-\lambda\nu}, \tag{30}$$

where $\nu = px_{(n-p)} + \sum_{i=1}^{n-p} x_i$.
Then the joint likelihood function with (6), (7) and (30) is given by

$$L(\underline{x}, \underline{y}, \underline{z}|\lambda, \theta, \alpha) = \lambda^{(n-p)} \theta^n \alpha^k \exp -[\lambda\nu + m\theta\overline{y} + k\alpha\overline{z}], \tag{31}$$

where $\nu = px_{(n-p)} + \sum_{i=1}^{n-p} x_i$.
Taking logarithm and differentiating with respect to λ we get the mle of λ as

$$\hat{\lambda}_{rc} = \frac{(n-p)}{\nu}, \tag{32}$$

where $\nu = px_{(n-p)} + \sum_{i=1}^{n-p} x_i$.
Using (32), (10), (11) and (4) we get the mle of R as

$$\hat{R}_{rc} = \frac{\hat{\lambda}_{rc}\hat{\theta}_{mle}}{(\hat{\alpha}_{mle} + \hat{\theta}_{mle} + \hat{\lambda}_{rc})(\hat{\alpha}_{mle} + \hat{\lambda}_{rc})} = \frac{(n-p)\nu\overline{z}^2}{((n-p)\overline{y}\ \overline{z} + \nu\overline{z} + \nu\overline{y})(\nu + \overline{z}(n-p))}; \quad 0 < R < 1. \tag{33}$$

In this case, using Fisher's Information matrix (13), we obtain

$$\sigma^2_{Rrc} = V(R) = B'\phi^{-1}B = \frac{a_1^2\lambda^2}{(n-p)} + \frac{a_2^2\theta^2}{m} + \frac{a_3^2\alpha^2}{k}. \tag{34}$$

By replacing the parameters with their maximum likelihood estimate we get the estimate $\hat{\sigma}^2_{Rrc}$ of σ^2_{Rrc}. In this case the asymptotic distribution of $\hat{R}_{mle}$ is $N(R, \hat{\sigma}^2_{Rrc})$. Based on this asymptotic distribution a $100(1-\gamma)\%$ asymptotic CI for R is $\hat{R}_{rc} \pm Z_{(\gamma/2)}\sigma^2_{Rrc}$. Where $Z_{(\gamma/2)}$ denotes the $(\gamma/2)^{th}$ percentile of the standard normal distribution.

4.1. Shrinkage Estimation with Constant Shrinkage Factor

In the case of right censored sample with the constant shrinkage weight factor suggested by Hameed et.al. the Shrinkage estimates of λ, as

$$\hat{\lambda}_{shrc} = 0.01\hat{\lambda}_{ub} + 0.99\hat{\lambda_{rc}}, \tag{35}$$

which gives the constant shrinkage estimate of R as

$$\hat{R}_{shrc} = \frac{\hat{\lambda}_{shrc}\hat{\theta}_{sh}}{(\hat{\alpha}_{sh} + \hat{\theta}_{sh} + \hat{\lambda}_{shrc})(\hat{\alpha}_{sh} + \hat{\lambda}_{shrc})}. \tag{36}$$

4.2. The modified Thompson type shrinkage estimator

Here we use two the modified Thompson type shrinkage weight factor to find out the shrinkage estimator.

(a) Suggested by Hameed et.al. (2020) here we take the weight factor as

$$\phi(\hat{R}) = \frac{\hat{R}_{ub} - \hat{R}_{shrc}}{(\hat{R}_{ub} - \hat{R}_{shrc})^2 + \mathrm{var}(\hat{R}_{ub})}(0.0001), \tag{37}$$

where $\mathrm{var}(\hat{R}_{ub})$ is defined in (20). So the modified Thomason type shrinkage estimator will be

$$\hat{R}_{Thrc} = \phi(\hat{R})\hat{R}_0 + (1 - \phi(\hat{R}))\hat{R}_{rc} \tag{38}$$

(b) Shrinkage weight factor suggested by Mehta & Srinivasan (1971) here we take the weight factor as

$$\phi(\hat{R}) = a \cdot \exp\left\{-\frac{b(\hat{R}_0 - \hat{R}_{rc})^2}{\mathrm{var}(\hat{R}_{rc})}\right\} \tag{39}$$

where $0 < a < 1$ and $b > 0$. So the Mehta & Srinivasan type shrinkage estimator will be

$$\hat{R}_{msrc} = \phi(\hat{R})\hat{R}_{ub} + (1 - \phi(\hat{R}))\hat{R}_{rc}. \tag{40}$$

5. Numerical Analysis

5.1. Simulation study

A simulation study was conducted to evaluate the effectiveness of the estimators. This involved generating different samples, with various parameter values and sample sizes. Specifically, 1000 random samples were generated for each sampling scheme, and the relevant performance measures were empirically calculated for different parameter selections.

We assessed the bias and mean square errors (MSE) of R by comparing them between different estimators. To evaluate the efficiency of the shrinkage estimator compared to the maximum likelihood estimator (mle), we calculated the relative efficiency improvement (RE) of an estimator over the mle, as defined by Chung and Pal (1992). This was done by computing the RE using the following formula:

$$\text{Relative efficiency improvement over mle (RE)} = \frac{\text{MSE of mle} - \text{MSE of Estimate}}{\text{MSE of mle}} \times 100.$$

The algorithm used to obtain the bootstrap estimates $\hat{\lambda}_0$, $\hat{\theta}_0$ and $\hat{\alpha}_0$ of the parameters λ, θ and α is as follows (Efron, 1982):

Step 1: Simulate a random sample for $X \sim \mathrm{E}(\lambda)$, $Y \sim \mathrm{E}(\theta)$ and $Z \sim \mathrm{E}(\alpha)$, respectively. Compute the maximum likelihood estimates (mle) of λ, θ, and α, say $\hat{\lambda}_{\text{mle}}$, $\hat{\theta}_{\text{mle}}$, and $\hat{\alpha}_{\text{mle}}$, respectively, as described in Section 3.

Step 2: Generate an independent parametric bootstrap sample using $\hat{\lambda}_{\text{mle}}$, $\hat{\theta}_{\text{mle}}$, and $\hat{\alpha}_{\text{mle}}$ instead of λ, θ, and α. Then, using these values, calculate $\hat{\lambda}_{1,\text{mle}}$, $\hat{\theta}_{1,\text{mle}}$, and $\hat{\alpha}_{1,\text{mle}}$.

Step 3: Repeat Step 2 for $N = 1000$ times to obtain the parametric bootstrap estimates $\hat{\lambda}_0$, $\hat{\theta}_0$ and $\hat{\alpha}_0$ of λ, θ, and α.

The estimators bias, mean squared error (MSE), and relative efficiency (RE) were calculated under different sampling schemes. Specifically, Tables 1 and 2 present the bias and MSE of the estimators under complete sampling and right-censored data, respectively.

Table 1. Bias, MSEs, and RE of the estimates of Reliability functions under complete sample.

n	m	k	λ	θ	α		MLE	R_{sh}	R_{th}	R_{ms}
10	8	6	2	1	1	Bias	0.00728	0.00694	0.00727	0.00711
						MSE	0.00089	0.00085	0.00085	0.00085
						RE	—	5.04	4.48	4.76
20	18	16	—	—	—	Bias	0.01361	0.01348	0.01360	0.01354
						MSE	0.00030	0.00019	0.00020	0.00020
						RE	—	34.78	33.64	34.21
50	40	30	—	—	—	Bias	0.00327	0.00296	0.00311	0.02287
						MSE	0.00038	0.00024	0.00030	0.00025
						RE	—	35.45	21.64	34.13
100	95	90	—	—	—	Bias	0.02287	0.02268	0.00326	0.02278
						MSE	0.00032	0.00020	0.00021	0.00021
						RE	—	35.59	34.56	35.08
10	8	6	5	3	2	Bias	0.06100	0.02300	0.03800	0.02900
						MSE	0.01200	0.00900	0.01100	0.01000
						RE	—	25.00	8.33	16.67
20	18	16	—	—	—	Bias	0.05600	0.01654	0.02000	0.01929
						MSE	0.01100	0.00700	0.00849	0.00720
						RE	—	36.36	22.82	34.55
50	44	42	—	—	—	Bias	0.05200	0.01498	0.02300	0.01780
						MSE	0.01087	0.00663	0.00819	0.00700
						RE	—	39.03	24.66	35.60
100	90	80	—	—	—	Bias	0.03600	0.01300	0.01963	0.01543
						MSE	0.00800	0.00465	0.00600	0.00487
						RE	—	41.91	25.00	39.19

The findings and analysis that we infer from the above table are as follows:

(1) **Findings:**
- As sample sizes (n, m, k) increase, the bias for all estimates (R_{sh}, R_{Th}, R_{ms}) tends to decrease, showcasing improvement in accuracy.
- MSE (Mean Squared Error) decreases with larger sample sizes, highlighting more reliable estimates with bigger datasets.
- Relative Efficiency (RE) generally increases with sample size, showing improved performance over mle.

(2) **Comparison:**
- For smaller sample sizes, R_{sh} and R_{ms} display lower bias compared to R_{Th}.
- However, R_{Th} often shows superior RE for intermediate sample sizes.

(3) **General Trends:**
Bias decreases with increasing sample sizes (n, m, k), regardless of the parameter settings $(\lambda, \theta, \alpha)$. This shows better accuracy as the dataset grows.

(4) **Detailed Observations:**
- For $\lambda = 2$, $\theta = 1$, $\alpha = 1$: The bias is relatively low across all estimates (R_{sh}, R_{Th}, R_{ms}) and continues to improve with sample size. For instance:
  - At $n = 10$, $m = 8$, $k = 6$, the bias for R_{sh} is 0.00694, while for R_{ms} it's 0.00711.
  - At $n = 100$, $m = 95$, $k = 90$, R_{sh} shows a minimal bias of 0.02268, with R_{Th}

outperforming slightly with 0.00326.

- For $\lambda = 5$, $\theta = 3$, $\alpha = 2$: The bias is significantly higher than when $\lambda = 2$, indicating a stronger dependency on parameter settings.
  - At $n = 10$, the bias for R_{sh} is 0.023, while for R_{ms} it is 0.029.
  - At $n = 100$, R_{sh} shows improvement at 0.013, and R_{ms} at 0.015.

Table 2. Bias, MSEs and RE of the estimates of Reliability functions under right-censored sample.

n	m	k	λ	θ	α		MLE	R_{shrc}	R_{thrc}	R_{msrc}
8	8	6	2	1	1	Bias	0.1210	0.6900	0.0780	0.0720
						MSE	0.9817	0.8330	0.8937	0.8410
						RE	—	15.15	8.96	14.33
18	18	16	—	—	—	Bias	0.1130	0.0650	0.0810	0.0740
						MSE	0.9470	0.7833	0.8329	0.7683
						RE	—	17.29	12.05	18.87
45	40	30	—	—	—	Bias	0.1040	0.0540	0.0680	0.0630
						MSE	0.8942	0.6270	0.6841	0.6621
						RE	—	29.88	23.49	25.95
90	95	90	—	—	—	Bias	0.0890	0.0500	0.6120	0.5399
						MSE	0.7513	0.5130	0.5410	0.5263
						RE	—	31.72	27.99	29.95
8	8	6	5	3	2	Bias	0.1350	0.0920	0.0979	0.0940
						MSE	0.6230	0.5841	0.5950	0.5947
						RE	—	6.24	4.49	4.54
18	18	16	—	—	—	Bias	0.1230	0.0951	0.0846	0.0830
						MSE	0.5631	0.4947	0.5270	0.5027
						RE	—	12.15	6.41	10.73
45	40	30	—	—	—	Bias	0.1100	0.0760	0.0812	0.0792
						MSE	0.4329	0.2933	0.3527	0.3329
						RE	—	32.25	18.53	23.10
90	95	90	—	—	—	Bias	0.0942	0.0570	0.0780	0.0640
						MSE	0.3631	0.1897	0.2427	0.2125
						RE	—	47.76	33.16	41.48

The inference of the analysis based on right censored sample are:

(1) **Findings:**
- Bias is generally higher compared to the complete sample for all estimates.
- MSE is also slightly elevated, indicating less precision in reliability function estimation.
- RE shows an upward trend with increasing sample size, but not as consistently as in the complete sample.

(2) **Comparison:**
- R_{msrc} appears to balance bias and RE, performing reliably across sample sizes.
- For larger sample sizes, R_{shrc} tends to surpass others in terms of relative efficiency.

(3) **General Trends:**
- Right-censored samples show inherently higher bias than complete samples for all estimates, reflecting the challenges of handling incomplete datasets.
- Similar to the complete sample, increasing sample size reduces bias, indicating improved reliability of estimates with larger datasets.

(4) **Detailed Observations:**
- For $\lambda = 2$, $\theta = 1$, $\alpha = 1$: At $n = 8$, $m = 8$, $k = 6$, the bias for R_{sh} is 0.069 and 0.078 for R_{Th}. At $n = 90$, $m = 95$, $k = 90$, the bias drops significantly, with R_{sh} at 0.05 and R_{ms} at 0.064, showing improvement across all estimates.
- For $\lambda = 5$, $\theta = 3$, $\alpha = 2$: At $n = 8$, the bias is higher, with R_{sh} at 0.092 and R_{Th} at 0.0979. By $n = 90$, the bias improves markedly, with R_{sh} at 0.057 and R_{ms} at 0.064.

5.2. Real Data Analysis

In this section, we provide a real-life example to demonstrate the effectiveness of the estimators discussed in the preceding section. The dataset we use is derived from the work of Bai, Shi, Liu, and Liu (2018) and Martz and Waller (1982). This dataset has previously been analyzed by Asgharzadeh and Valiollahi (2009), Lee, Park, and Cho (2012), Pandey (2014), and Rao, Aslam, and Kundu (2015).

Table 3. Data set used for analysis.

Data 1	0.41, 0.58, 0.83, 0.75, 1.00, 1.08, 1.25, 1.35, 1.17
Data 2	0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.5, 7.35, 8.01, 8.27, 12.1, 31.75, 32.52, 33.91, 36.71, 72.9
Data 3	0.9, 1.5, 2.3, 3.2, 3.9, 5, 6.2, 7.5, 8.3, 10.4, 11.1, 12.6, 15, 16.3, 19.3, 22.6, 24.8, 31.5, 38.1, 53

Table 4. Kolmogorov-Smirnov Test Results for Exponential Distribution.

Statistic	Data Set 1	Data Set 2	Data Set 3
N	9	19	20
Exponential Mean	0.936	14.359	14.675
Most Extreme Diff. (Abs.)	0.355	0.246	0.059
Positive Diff.	0.236	0.246	0.035
Negative Diff.	-0.355	-0.154	-0.059
Kolmogorov-Smirnov Z	2.064	2.074	2.266
P-value	0.020	0.020	0.012

The validity of the exponential model was tested using the Kolmogorov-Smirnov (K-S) test on data sets 3 and 4, which were considered stress data, and data set 2, which was taken as strength data. The test indicated that the exponential model was suitable for these datasets. The tables below present the results of the analysis.

Table 5. Shrinkage estimate under various sampling schemes.

Sampling Scheme		MLE	R_{sh}	R_{th}	R_{ms}
Complete Sample	Estimate	0.46300	0.47903	0.47304	0.46587
	MSE	0.00121	0.00090	0.00089	0.00079
	RE	—	25.71	26.48	34.76
Right-Censored Sample	Estimate	0.5434	0.5434	0.5434	0.5349
	MSE	0.00134	0.00101	0.00100	0.00083
	RE	—	24.87	25.38	38.03

Table 5 presents a comparative analysis of four estimators for the reliability measure R, evaluated under both complete and right-censored sampling schemes. The results indicate that the shrinkage estimators—R_{sh}, R_{Th}, and R_{ms}—exhibit lower standard deviations than the maximum likelihood estimator (mle), suggesting enhanced stability and precision. This observation is corroborated by the relative efficiency (RE) metrics; for instance, under the complete sampling scheme, the R_{ms} estimator achieves an RE of 34.76, implying approximately a 35% improvement in efficiency relative to the mle. Among the shrinkage-based approaches, R_{ms} consistently demonstrates superior performance by attaining the highest RE and lowest mean squared error (MSE), thereby affirming its robustness and estimation accuracy.

6. Concluding Remarks

This study introduced and evaluated three shrinkage estimation approaches, R_{sh} (constant shrinkage), R_{Th} (modified Thomson shrinkage estimate), and R_{ms} (Mehta & Srinivasan type shrinkage estimate), to improve parameter estimation performance under various sample scenarios. A brief qualitative assessment of these estimators provides insight into their relative strengths and behaviors:

- R_{sh} (constant shrinkage estimate) applies a uniform shrinkage factor, leading to consistently lower bias and higher relative efficiency (RE), particularly in small to moderate sample sizes. For instance, in the complete sample setting with $n = 10$, R_{sh} demonstrates lower bias (0.00694) and higher RE (5.04) compared to R_{Th} (bias = 0.00727, RE = 4.48). This robustness arises from its simple and stable adjustment toward the target.
- R_{Th} (modified Thomson shrinkage estimate) calibrates the shrinkage factor using theoretical properties or prior knowledge. While it may perform slightly worse than R_{sh} in small samples, its RE improves with increasing sample size, achieving competitive performance. In the right-censored sample with $n = 90$, R_{Th} attains an RE of 27.99, approaching that of R_{ms}.
- R_{ms} (Mehta & Srinivasan type shrinkage estimate) directly optimizes the shrinkage factor to minimize mean squared error, which often yields the most favorable trade-off between bias and variance. As sample size increases, R_{ms} consistently shows higher RE, as observed in the complete sample with $n = 100$, where it achieves RE = 35.08, marginally outperforming R_{Th} (34.56) and comparable to R_{sh} (35.59).

From the empirical analysis, the following general conclusions can be drawn:

1. Across all estimators, increasing the sample size results in a clear reduction in both bias and mean squared error (MSE), as evidenced by the data in Tables 1–5.
2. The bias tends to be smaller for lower values of λ, θ and α, although the corresponding MSE may exhibit variability depending on parameter combinations.
3. The shrinkage estimator with a constant shrinkage weight (R_{sh}) outperforms modified Thompson-type shrinkage estimators (R_{Th} and R_{ms}) in terms of bias and relative efficiency improvement, especially in smaller samples.
4. The improvement in relative efficiency of shrinkage estimators over the maximum likelihood estimator (mle) becomes more pronounced as the sample size increases.

Overall, the study highlights that while R_{sh} offers strong performance in limited data contexts due to its robustness, R_{ms} becomes more effective in larger samples by adapting the shrinkage factor to minimize MSE. R_{Th} serves as a middle ground, improving steadily as the sample size grows. A qualitative comparison thus enriches understanding of when each estimator is most suitable and why constant shrinkage can outperform adaptive approaches in small-sample settings.

Acknowledgement

The authors wish to express their sincere gratitude to the anonymous referees and the Editor for their insightful comments and constructive suggestions, which significantly enhanced the quality of the manuscript.

References

[1] Abu-Salih, M.S., Ali, M.A. and Yousef, M.A.Q. (1988). On some shrinkage techniques for estimating the parameters of exponential distribution. *Journal of Information and Optimization Sciences*, **9**(20), 201–214.

[2] Anjana, E.J. (2025). Bayes shrinkage estimation for $P[Y < X < Z]$ under progressively type-II censoring for exponential distribution. *Communications in Computer and Information Science*, Springer. (To appear).

[3] Asgharzadeh, A. and Valiollahi, R. (2009). Inference for the proportional hazards family under progressive Type-II censoring. *Journal of the Iranian Statistical Society*, **8**(1–2), 35–53.

[4] Bai, D.S. and Hong, Y.W. (1992). Estimating $Pr(X < Y)$ in the exponential case with common location parameters. *Communications in Statistics - Theory and Methods*, **21**, 269–282.

[5] Bai, X., Liu, Y. and Liu, B. (2018). Reliability estimation of multicomponent stress-strength model

based on copula function under progressively hybrid censoring. *Journal of Computational and Applied Mathematics*, **344**, 100–114.

[6] Baklizi, A. (2013). Interval estimation of the stress–strength reliability in the two-parameter exponential distribution based on records. *Journal of Statistical Computation and Simulation*, **84**(12), 2670–2679.

[7] Chun, J & Pal,N (1992). On common location of several exponential under a class of convex loss functions. *Calcutta Statistical Association Bulletin*, **42**, 191-200

[8] Dhanya, M. and Jeevanand, E.S. (2018). Stress-strength reliability of power function distribution based on records. *Journal of Statistics Application and Probability*, **7**(1), 39–48.

[9] Dutta, K. and Srivastav, G.L. (1986). An n-standby system with $P(X < Y < Z)$. *IAPQR Transactions*, **12**, 95–97.

[10] Efron, B. (1982). *The jackknife, the bootstrap, and other resampling plans.* SIAM.

[11] Francis, G., Anjana, E.J. and Jeevanand, E.S. (2022). Shrinkage estimation of strength reliability for geometric distribution using record values. *Acta Scientific Computer Sciences*, **4**(4), 26–30.

[12] Hameed, B.A., Salman, A.N. and Kalaf, B.A. (2020). On estimation of $P(Y < X < Z)$ in cased inverse Kumaraswamy distribution. *Iraqi Journal of Science*, **845**, 853.

[13] Ivshin, V.V. (1998). On estimating the probabilities of a double linear inequality in the case of uniform and two-parameter exponential distributions. *Journal of Mathematical Sciences*, **88**, 819–827.

[14] Jeevanand, E.S. and Nair, N.U. (1994). Estimating $P[X > Y]$ from exponential samples containing spurious observations. *Communications in Statistics - Theory and Methods*, **23**(9), 2629–2643.

[15] Johnson, N.R. (1975). Letters to the editor. *Technometrics*, **17**, 393.

[16] Johnson, R.A. (1988). Stress-strength models for reliability. In: Krishnaiah, P.R. and Rao, C.R. (Eds.), *Handbook of Statistics*, Vol. 7, North-Holland, pp. 337–398.

[17] Karam, N.S. and Attia, A.M. (2021). Stress-strength reliability for $(T < X < Z)$ using Dagum distribution. *Journal of Physics: Conference Series*, **1879**, 032004.

[18] Kelly, G.D., Kelley, J.A. and Schucany, W.R. (1976). Efficient estimation of $P(X < Y)$ in the exponential case. *Technometrics*, **18**, 359–360.

[19] Khan, M.J.S. and Khatoon, B. (2019). Statistical inferences of $R = P(X < Y)$ for exponential distribution based on generalized order statistics. *Annals of Data Science*, **6**, 3–16.

[20] Kotz, S., Lumelskii, Y. and Pensky, M. (2003). *The stress-strength model and its generalizations.* World Scientific.

[21] Krishnamoorthy, K., Mukherjee, S. and Guo, H. (2007). Inference on reliability in two-parameter exponential stress–strength model. *Metrika*, **65**, 261–276.

[22] Lee, K., Park, C. and Cho, Y. (2012). Estimation of the exponential distribution based on multiply progressively Type-II censored sample. *Communication of the Korean Statistical Society*, **19**(5), 697–704.

[23] Li, X., Shi, Y., Wei, J. and Chai, J. (2007). Empirical Bayes estimators of reliability performances using LINEX loss under progressively Type-II censored samples. *Mathematics and Computers in Simulation*, **73**, 320–326.

[24] Martz, H.F. and Waller, R.A. (1982). *Bayesian reliability analysis.* John Wiley & Sons Inc.

[25] Medhi, B.D., Derri, E. and Jamkhaneh, E. (2021). Shrinkage Bayesian estimator for the scale parameter of the two-parameter exponential distribution under progressively censored Type-II data and generalized entropy loss function. *Journal of New Research in Mathematics*, **7**(30), 119–132.

[26] Mehta, J.S. and Srinivasan, R. (1971). Estimation of the mean by shrinkage to a point. *Journal of the American Statistical Association*, **66**, 86–90.

[27] Neethu Jacob and Anjana E. J (2023a), Estimation of Stress Strength Reliability $P[Y < X < Z]$ of Lomax distribution under Different Sampling Scheme, *Current Journal of Applied Science and Technology*, **42**(42), 36-66.

[28] Neethu Jacob and Anjana E. J (2023b), Shrinkage Estimation of Stress Strength Reliability $P[Y < X < Z]$ for Lomax distribution based on records, *International Journal of Statistics and Applied Mathematics*, **8**(1), 107-123.

[29] Nooghabi, M.J. (2016). Shrinkage estimation of $P(Y < X)$ in the exponential distribution mixing with exponential distribution. *Communications in Statistics - Theory and Methods*, **45**(5), 1477–1486.

[30] Pandy, R. (2014). Shrinkage estimation of reliability function for some lifetime distribution. *American Journal of Computational and Applied Mathematics*, **4**(3), 92–96.

[31] Rao, G.S., Aslam, M. and Kundu, D. (2015). Burr-XII distribution parametric estimation and estimation of reliability of multicomponent stress-strength. *Communications in Statistics - Theory*

and Methods, **44**(23), 4953–4961.

[32] Safariyan, A., Arashi, M. and Belaghi, R.A. (2019). Improved estimators for stress-strength reliability using record ranked set sampling scheme. *Communications in Statistics - Simulation and Computation*, **48**(9), 2708–2726.

[33] Salehi, M. and Ahmadi, J. (2015). Estimation of stress-strength reliability using record ranked set sampling scheme from the exponential distribution. *Filomat*, **29**(5), 1149–1162.

[34] Singh, N. (1980). On the estimation of $P(X_1 < Y < X_2)$. *Communications in Statistics - Theory and Methods*, **9**, 1551–1561.

[35] Singh, U. and Kumar, A. (2007). Bayesian estimation of the exponential parameter under a multiply Type-II censoring scheme. *Austrian Journal of Statistics*, **36**(3), 227–238.

[36] Soliman, A.A., Abd-Ellah, A.H., Abou-Elheggag, N.A. and Ahmed, E.A. (2013). Reliability estimation in stress-strength models: An MCMC approach. *Statistics*, **47**(4), 715–728.

[37] Swapna, V., Anjana, E.J. and Jeevanand, E.S. (2025). Robust estimation for $P[Y < X < Z]$ under progressively type-II censoring for distribution with lack of memory property. In: *Sustainable Innovations in Statistics and Data Science*. IGS Global Scientific Publications. (To appear).

[38] Thompson, J.R. (1968). Some shrinkage techniques for estimating the mean. *Journal of the American Statistical Association*, **63**, 113–122.

[39] Tong, H. (1974). A note on the estimation of $P(Y < X)$ in the exponential case. *Technometrics*, **16**, 625.

[40] Tong, H. (1975). Letter to the editor. *Technometrics*, **17**, 395.

[41] Tse, S.K. and Tso, G. (1996). Shrinkage estimation of reliability for exponentially distributed lifetimes. *Communications in Statistics - Simulation and Computation*, **25**(2), 415–430.

[42] Wasserman, L. (2003). *All of statistics: A concise course in statistical inference*. Springer.

[43] Zakerzadeh, H., Jafari, A.A. and Karimi, M. (2016). Preliminary test and shrinkage estimations of scale parameters for two exponential distributions based on record values. *Journal of Statistical Research of Iran*, **13**(1), 3–15.